Published by Institute of Physics Publishing for SISSA

Received: *October 17, 2005*
Revised: *January 10, 2006*
Accepted: *February 2, 2006*
Published: *February 17, 2006*

Renormalizability of noncommutative $SU(N)$ gauge theory

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Abstract: We analyze the renormalizability properties of pure gauge noncommutative $SU(N)$ theory in the θ-expanded approach. We find that the theory is one-loop renormalizable to first order in θ.

Keywords: Renormalization Regularization and Renormalons, Non-Commutative Geometry.

JHEP02(2006)046

Contents

1. Introduction

The early motivation to introduce noncommutativity of coordinates was the search for invariant regulator in field theory. Many other reasons for studying noncommutative manifolds have appeared since; still, the renormalizability of field theories, on noncommutative Minkowski space for example, has not been established. The aim of this paper is to add further results to this discussion.

To introduce the problem and the notation, we give a couple of basic definitions. Noncommutative Minkowski space is the algebra generated by coordinates $\hat{x}^\mu$, $\mu = 0, 1, 2, 3$, which obey the commutation relations of canonical type:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} = \text{const.} \tag{1.1}$$

The algebra (1.1) can be represented by the algebra of functions on commutative four-dimensional manifold with the Moyal-Weyl product as multiplication. The latter is defined as

$$\phi(x) \star \chi(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\phi(x)\chi(y)|_{y\to x}\,. \tag{1.2}$$

The fields are functions of coordinates; for example, the vector potential $\hat{A}_\mu(x)$ of the gauge group $U(N)$ is, in this context, defined by

$$\hat{A}_\mu(x) = \hat{A}^A_\mu(x)\mathbf{t}^A, \tag{1.3}$$

where $\mathbf{t}^A$ are ordinary $N \times N$ matrices, generators of $U(N)$. We denote the generators of $SU(N)$ by t^a: the capital letters denote the $U(N)$ indices, while the small letters denote the $SU(N)$ indices, $\mathbf{t}^A \in \{I, t^a\}$. The field strength transforms in the adjoint representation; it is given by

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i(\hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu). \tag{1.4}$$

The generators of $SU(N)$ satisfy

$$[t^a, t^b] = if^{abc}t^c, \qquad \{t^a, t^b\} = d^{abc}t^c, \tag{1.5}$$

f^{abc} are the structure constants of $SU(N)$, d^{abc} are the symmetric symbols, $d^{abc} = \mathrm{Tr}\,\{t^a, t^b\}t^c$: we use the normalization $\mathrm{Tr}\,(t^a t^b) = \delta^{ab}$. For gauge groups different from $U(N)$ the commutator term in (1.4) does not take values in the Lie algebra of the group, as the $\star$-product is not commutative. It is clear therefore that not all gauge groups can be realized on the space (1.1) in the described manner.

As the gauge fields are represented by functions on $\mathbb{R}^4$, the integration and the action can be defined straightforwardly. The action for the pure gauge theory reads

$$S = -\frac{1}{4}\,\mathrm{Tr}\int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} = -\frac{1}{4}\,\mathrm{Tr}\int d^4x \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}, \tag{1.6}$$

the last equality is valid due to properties of the Moyal-Weyl multiplication. One can easily include matter fields. This basically concludes the definition of the classical theory. Compared to the commutative gauge theory, it has new, specific features. For quantization, one usually takes $\theta^{0i} = 0$ as in that case there are no temporal derivatives of higher order in the lagrangian. This means that the generalized momenta are the same as in the classical theory and the momentum dependence in the path integral is Gaussian. Stated differently, $\theta^{0i} = 0$ provides the unitarity. In perturbation expansion Feynman rules get modified in accordance with the definition of $\star$-multiplication. The most important result concerning renormalization is the mixing of ultraviolet and infrared divergencies: in higher-order diagrams they are entangled in such a way that no efficient renormalization procedure can be defined. This phenomenon is typical for noncommutative theories and has been thoroughly discussed for ϕ^4 theory, [1]. See also the alternative heat-kernel derivation in [2]. The existence of gauge symmetry does not remove UV/IR mixing: for more detailed analysis for $U(1)$ we refer to [3], for nonabelian theories to [4]. Consequently, the status of renormalizability of noncommutative gauge theories is the same as for noncommutative scalar field theory: in general, the theories are not renormalizable.

A different representation of noncommutative gauge theories on commutative $\mathbb{R}^4$ was developed in [5]. Its main idea is to enlarge the basis of the algebra to the enveloping algebra of the group. Although only the infinitesimal gauge transformations can be defined in this way, the construction goes for arbitrary gauge groups and their tensor products. As it was shown in [5], every physical field can written as a formal expansion in noncommutativity parameter $\theta^{\mu\nu}$; the leading term in the expansion of noncommutative field is its commutative counterpart. Therefore, for $\theta^{\mu\nu} = 0$ noncommutative theory reduces to the usual, commutative one: the representation is a deformation of the commutative theory. This fact can also be related to the Seiberg-Witten result [6] that classically noncommutative and commutative gauge theories are equivalent. The expansions of the vector potential and the field strength to linear order in θ read

$$\begin{aligned}
\hat{A}_\rho(x) &= A_\rho(x) - \frac{1}{4}\,\theta^{\mu\nu}\,\{A_\mu(x), \partial_\nu A_\rho(x) + F_{\nu\rho}(x)\} + \dots \\
\hat{F}_{\rho\sigma}(x) &= F_{\rho\sigma}(x) + \frac{1}{2}\theta^{\mu\nu}\{F_{\mu\rho}(x), F_{\nu\sigma}(x)\} - \frac{1}{4}\theta^{\mu\nu}\{A_\mu(x), (\partial_\nu + D_\nu)F_{\rho\sigma}(x)\} + \dots,
\end{aligned} \tag{1.7}$$

and, as it was noted in [7], are not unique. The gauge field action (1.6) can be expanded, too. The parameter θ is treated as a coupling constant and thus propagators of all fields are defined as in the commutative theory. It can be shown that each of the interaction terms is invariant to commutative gauge transformations; noncommutative gauge symmetry is recorded only in the complete sum (1.7).

Obviously, a drawback of the 'θ-expanded' approach is that the results are necessarily expressed in powers of θ and calculated to a certain order: in practice, at most to second order. Thus one cannot obtain results which are nonanalytic in θ: and UV/IR mixing is an effect proportional to θ^{-1}. On the other hand, it does make very good sense to expand in θ if one wants to compare with the experiment: if exists, noncommutativity is very small. The question of renormalizability can be approached as well; in particular, negative results which one might obtain can be regarded conclusive. Let us stress that, in principle, there is no reason to expect to get the same results as for the nonexpanded theory, as the two representations of the gauge symmetry are different; in fact, $SU(N)$ cannot even be defined in the nonexpanded representation. It was shown in [8 – 10] that $U(1)$ and $SU(2)$ gauge theories with fermions are not renormalizable: the divergencies which cannot be removed exist both at θ-linear and θ-quadratic level. The results concerning pure gauge theories on the other hand, are somewhat partial: for example, the photon propagator in $U(1)$ theory is renormalizable in a generalized sense, to linear and possibly to all orders, [11]. Similar statement holds for the gluon propagator in $SU(2)$ to linear order, [10]. In this paper we investigate renormalizability of the first order-corrected $SU(N)$ theory: as we shall see from the form of divergencies, theory is in this order renormalizable. This result, we think, keeps the discussion on the renormalizability of field theories on noncommutative Minkowski space open.

The plan of the paper is the following. In the next section we expand the classical action to second order in quantum fields and do the path integral quantization. In the third section we present the result of the calculation of divergencies in the one-loop effective action and we show that the action can be renormalized in a very simple way. A summary of the results is given in the concluding section.

2. Expansion in background fields

The classical action for the gauge field expanded to first order in noncommutativity is [5]:

$$\begin{aligned} S_{cl} &= -\frac{1}{4}\mathrm{Tr}\int d^4x \hat{F}_{\mu\nu}\star\hat{F}^{\mu\nu} \\ &= \mathrm{Tr}\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{8}\theta^{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}-\frac{1}{2}\theta^{\mu\nu}F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma}\right) \\ &= \int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}+\frac{1}{16}\theta^{\mu\nu}d^{abc}(F^a_{\mu\nu}F^b_{\rho\sigma}F^{\rho\sigma c}-4F^a_{\mu\rho}F^b_{\nu\sigma}F^{\rho\sigma c})\right). \end{aligned} \tag{2.1}$$

We are dealing with the gauge group $SU(N)$: $a,b,c = 1,\ldots,N^2-1$ are the group indices. The group metric is Euclidean so there is no need to distinguish between upper and lower indices. The gauge fields are assumed to be in the adjoint representation; we

will comment further on this restriction. We quantize the theory by functional integration of the vector potential. The integration is done around the classical configuration, i.e., we use the background field method to find the effective action. The main contribution to the integral is given by the Gaussian integral: to find it, technically, we need to write the vector potential in the action as a sum of the background part A^a_μ and the quantum fluctuation $\mathcal{A}^a_\mu$, and find the part quadratic in $\mathcal{A}^a_\mu$. By this we determine the second functional derivative of the classical action with respect to the vector potential. In the saddle-point approximation, the result of the functional integration is

$$\Gamma[A^a_\mu] = S[A^a_\mu] + \frac{i}{2}\log\det S^{(2)}[A^a_\mu]. \tag{2.2}$$

We are dealing with the gauge symmetry and therefore the gauge fixing term has to be included in the action as well:

$$S = S_{cl} + S_{gf}, \qquad S_{gf} = -\frac{1}{2}\int d^4x (D_\mu \mathcal{A}^{\mu a})^2, \tag{2.3}$$

with $D_\mu \mathcal{A}^a_\nu = \partial_\mu \mathcal{A}^a_\nu + f^{abc} A^b_\mu \mathcal{A}^c_\nu$. The one-loop effective action,

$$\Gamma^1[A^a_\mu] = \frac{i}{2}\log\det S^{(2)}[A^a_\mu] = \frac{i}{2}\mathrm{Tr}\,\log S^{(2)}[A^a_\mu], \tag{2.4}$$

can be obtained in the usual way, by perturbative expansion of the logarithm. As we already explained the method in details in [9], we discuss here only the points specific to the present calculation.

In the case we are studying, the quadratic part of the action is very complicated. Very involved too are the operator traces whose divergent parts we calculate by dimensional regularization. It is necessary therefore to develop a strategy already at this stage. We will calculate the divergencies at the special, constant value of the classical vector potential,

$$A^a_\mu = \text{const.} \tag{2.5}$$

At the end of the calculation the full expressions will be restored from covariance, replacing $f^{abc}A^b_\mu A^c_\nu$ by $F^a_{\mu\nu}$. The idea is not new; for a similar derivation see [13]. The use of background field method guarantees the covariance [14], as doing the path integral, we fix the local symmetry of the quantum field $\mathcal{A}^a_\mu$ while the gauge symmetry of the background field A^a_μ is manifestly preserved. Having in mind the dimension-regularization formulae, we see that in fact it is an advantage to deal with nonabelian theory, as the terms without derivatives are by far the simplest. At the moment, the assumption (2.5) means just that derivatives commute with A^a_μ in $S^{(2)}[A^a_\mu]$.

Extracting the quadratic part of the action is a straightforward but tedious calculation. The result has the form

$$S^{(2)} = \frac{1}{2}\mathcal{A}^a_\alpha \left[g^{\alpha\beta}\delta_{ab}\Box + (N_1 + N_2 + T_2 + T_3 + T_4)^{\alpha a,\beta b}\right]\mathcal{A}^b_\beta. \tag{2.6}$$

The operators N_1 and N_2 originate from the commutative action, while T_2, T_3 and T_4 denote interaction terms linear in θ. The index of the operator indicates the number of background fields A^a_μ which it contains. The operators are given by

$$(N_1)^{\alpha a,\beta b} = -2f^{abc}A^c_\mu g^{\alpha\beta}\partial^\mu = -2i(A_\mu)_{ab}g^{\alpha\beta}\partial^\mu, \tag{2.7}$$

where we introduced the matrix notation

$$(A_\mu)_{ab} = -if^{abc}A^c_\mu = A^c_\mu (T^c)_{ab}, \tag{2.8}$$

because the structure constants are the matrix elements of generators in the adjoint representation. The same notation will be used for the field strengths as it is very useful. For example,

$$(A_1 \dots A_{2n})_{ab} = (A_{2n} \dots A_1)_{ba}, \quad (A_1 \dots A_{2n+1})_{ab} = -(A_{2n+1} \dots A_1)_{ba}. \tag{2.9}$$

The rest of the vertices in (2.6) read

$$(N_2)^{ab}_{\alpha\beta} = -2f^{cab}F^c_{\alpha\beta} - (A_\mu A^\mu)^{ab} g_{\alpha\beta}, \tag{2.10}$$

$$\begin{aligned}
(T_2)^{ab}_{\alpha\beta} = \frac{1}{4} d^{abc} \big[& -2\theta_{\rho\alpha} F^c_{\mu\beta} \partial^\mu \partial^\rho - 2\theta_{\rho\beta} F^c_{\mu\alpha} \partial^\mu \partial^\rho - \theta^{\rho\sigma} F^c_{\rho\sigma} g_{\alpha\beta} \Box + \theta^{\rho\sigma} F^c_{\rho\sigma} \partial_\alpha \partial_\beta \\
& + 2\theta^\rho{}_\alpha F^c_{\mu\rho} \partial^\mu \partial_\beta + 2\theta^\rho{}_\beta F^c_{\mu\rho} \partial^\mu \partial_\alpha - 2\theta^\rho{}_\beta F^c_{\alpha\rho} \Box - 2\theta^\rho{}_\alpha F^c_{\beta\rho} \Box - 4\theta^{\rho\sigma} F^c_{\mu\rho} g_{\alpha\beta} \partial^\mu \partial_\sigma \\
& + 2\theta^{\rho\sigma} F^c_{\beta\rho} \partial_\alpha \partial_\sigma + 2\theta^{\rho\sigma} F^c_{\alpha\rho} \partial_\beta \partial_\sigma - 2\theta_{\alpha\sigma} F^c_{\mu\beta} \partial^\mu \partial^\sigma - 2\theta^\rho{}_\beta F^c_{\alpha\nu} \partial^\nu \partial_\rho \big],
\end{aligned}$$

$$\begin{aligned}
(T_3)^{ab}_{\alpha\beta} = -\frac{i}{4} d^{acd} \Big[& 2\theta_{\rho\alpha} F^d_{\mu\beta} (A^\mu)_{bc} \partial^\rho + 2\theta_{\rho\beta} F^d_{\mu\alpha} (A^\rho)_{bc} \partial^\mu \\
& + \theta^{\rho\sigma} g_{\alpha\beta} F^d_{\rho\sigma} (A^\mu)_{bc} \partial_\mu - \theta^{\rho\sigma} F^d_{\rho\sigma} (A_\alpha)_{bc} \partial_\beta \\
& - 2\theta^\rho{}_\alpha F^d_{\mu\rho} (A^\mu)_{bc} \partial_\beta + 2\theta^\rho{}_\alpha F^d_{\beta\rho} (A^\nu)_{bc} \partial_\nu \\
& + 2\theta^{\rho\sigma} g_{\alpha\beta} F^d_{\mu\rho} (A^\mu)_{bc} \partial_\sigma - 2\theta^{\rho\sigma} F^d_{\beta\rho} (A_\alpha)_{bc} \partial_\sigma - 2\theta^\rho{}_\beta F^d_{\mu\rho} (A^\alpha)_{bc} \partial^\mu \\
& + 2\theta^\rho{}_\beta F^d_{\alpha\rho} (A^\nu)_{bc} \partial_\nu + 2\theta^{\rho\sigma} g_{\alpha\beta} F^d_{\mu\rho} (A^\sigma)_{bc} \partial^\mu - 2\theta^{\rho\sigma} F^d_{\alpha\rho} (A_\sigma)_{bc} \partial^\beta \\
& - 2\theta_{\alpha\beta} F^d_{\mu\nu} (A^\nu)_{bc} \partial^\mu + 2\theta_{\alpha\sigma} F^d_{\mu\beta} (A^\sigma)_{bc} \partial^\mu + 2\theta_{\rho\beta} F^d_{\alpha\nu} (A^\nu)_{bc} \partial^\rho \\
& - 2\theta^{\rho\sigma} F^d_{\alpha\beta} (A_\sigma)_{bc} \partial_\rho - (a \leftrightarrow b, \alpha \leftrightarrow \beta) \Big],
\end{aligned}$$

and

$$\begin{aligned}
(T_4)^{ab}_{\alpha\beta} = \frac{1}{8} d^{ced} \big[& -4\theta_{\rho\alpha} F^e_{\mu\beta} (A^\rho)_{ad} (A^\mu)_{bc} - \theta^{\rho\sigma} g_{\alpha\beta} F^e_{\rho\sigma} (A^\mu)_{ad} (A_\mu)_{bc} \\
& + \theta^{\rho\sigma} F^e_{\rho\sigma} (A_\beta)_{da} (A_\alpha)_{cb} + 4\theta^\rho{}_\alpha F^e_{\mu\rho} (A_\beta)_{ad} (A^\mu)_{bc} \\
& - 4\theta^\rho{}_\alpha F^e_{\beta\rho} (A^\nu)_{ad} (A_\nu)_{bc} - 4\theta^{\rho\sigma} g_{\alpha\beta} F^e_{\mu\rho} (A_\sigma)_{ad} (A^\mu)_{bc} \\
& + 4\theta^{\rho\sigma} F^e_{\beta\rho} (A^\sigma)_{ad} (A_\alpha)_{bc} + 2\theta_{\alpha\beta} F^e_{\mu\nu} (A^\mu)_{ad} (A^\nu)_{bc} \\
& - 2\theta_{\alpha\rho} F^e_{\mu\beta} (A^\mu)_{ad} (A^\rho)_{bc} - 2\theta_{\rho\beta} F^e_{\alpha\nu} (A^\rho)_{ad} (A^\nu)_{bc} \\
& + 2\theta_{\rho\sigma} F^e_{\alpha\beta} (A^\rho)_{ad} (A^\sigma)_{bc} - 4\theta^\rho{}_\beta f^{cab} F^d_{\mu\rho} F^{\mu\ e}_{\alpha} + (a \leftrightarrow b, \alpha \leftrightarrow \beta) \Big] \\
& + \frac{1}{8} d^{ced} f^{cab} \Big[2\theta^{\rho\sigma} F^d_{\rho\sigma} F^e_{\alpha\beta} + \theta_{\alpha\beta} F^d_{\mu\nu} F^{\mu\nu e} \Big] \\
& - \frac{1}{4} \theta^{\rho\sigma} f^{cab} d^{cde} (F^d_{\alpha\rho} F^e_{\beta\sigma} - F^d_{\beta\rho} F^e_{\alpha\sigma}).
\end{aligned}$$

As already noted, they are written under the assumption (2.5).

3. Divergences

From (2.6) we read off the second functional derivative of the action:

$$(S^{(2)}[A^a_\mu])^{\alpha a,\beta b} = g^{\alpha\beta}\delta_{ab}\Box + (N_1 + N_2 + T_2 + T_3 + T_4)^{\alpha a,\beta b}. \tag{3.1}$$

In order to calculate the effective action (2.4), we expand the logarithm. Denoting $\mathcal{I}^{\alpha a,\beta b} = g^{\alpha\beta}\delta_{ab}$, we write

$$\begin{aligned}\mathrm{Tr}\,\log S^{(2)}[A^a_\mu] &= \mathrm{Tr}\,\log \mathcal{I}\Box + \mathrm{Tr}\,\log\left(\mathcal{I} + \Box^{-1}(N_1 + N_2 + T_2 + T_3 + T_4)\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{Tr}\,\left(\Box^{-1}(N_1 + N_2 + T_2 + T_3 + T_4)\right)^n,\end{aligned} \tag{3.2}$$

where in the last equality the infinite normalization constant is neglected. The terms which contribute to the divergent one-loop effective action in zero-th order are

$$\Gamma^0 = \frac{i}{2}\left(-\frac{1}{2}\mathrm{Tr}\,(\Box^{-1}N_2)^2 + \mathrm{Tr}\,((\Box^{-1}N_1)^2\Box^{-1}N_2) - \frac{1}{4}\mathrm{Tr}\,(\Box^{-1}N_1)^4\right); \tag{3.3}$$

so the divergent part is

$$\Gamma^0|_{\mathrm{div}} = \frac{5N}{3(4\pi)^2\epsilon}\int d^4x F^a_{\mu\nu}F^{\mu\nu a}. \tag{3.4}$$

However, the ghost contribution should also be taken into account. It is derived from the ghost action introduced as in [10, 12, 14]

$$\Gamma_{gh}|_{\mathrm{div}} = \frac{N}{6(4\pi)^2\epsilon}\int d^4x F^a_{\mu\nu}F^{\mu\nu a}. \tag{3.5}$$

Thus the sum of (3.4) and (3.5) constitutes the standard result of the commutative theory. Our goal is to calculate the θ-linear divergencies. As $\theta^{\mu\nu}$ has the length dimension 2 and $F^{\mu\nu}$ is of dimension -2, the terms of the following types are possible: θF^3, $\theta(DF)^2$, $\theta\epsilon F^3$, $\theta\epsilon(DF)^2$. All of them have one θ and six A's: thus from (3.2) we need to extract the terms which contain one of the vertices T_i and have the sum of indices equal to 6. The trace to be calculated is

$$\begin{aligned}\Gamma^1|_{\mathrm{div}} = \frac{i}{2}\big(&-\mathrm{Tr}\,[\Box^{-1}N_2\Box^{-1}T_4] + \mathrm{Tr}\,[(\Box^{-1}N_1)^2\Box^{-1}T_4] + \mathrm{Tr}\,[(\Box^{-1}N_2)^2\Box^{-1}T_2]\\ &+ \mathrm{Tr}\,[\Box^{-1}N_1\Box^{-1}N_2\Box^{-1}T_3] + \mathrm{Tr}\,[\Box^{-1}N_2\Box^{-1}N_1\Box^{-1}T_3] - \mathrm{Tr}\,[(\Box^{-1}N_1)^3T_3]\\ &- \mathrm{Tr}\,[(\Box^{-1}N_1)^2\Box^{-1}N_2\Box^{-1}T_2] - \mathrm{Tr}\,[\Box^{-1}N_2(\Box^{-1}N_1)^2\Box^{-1}T_2]\\ &- \mathrm{Tr}\,[\Box^{-1}N_1\Box^{-1}N_2\Box^{-1}N_1\Box^{-1}T_2] + \mathrm{Tr}\,[(\Box^{-1}N_1)^4\Box^{-1}T_2]\big).\end{aligned} \tag{3.6}$$

In fact, using Bianchi identities it can easily be seen that all possible combinations of the form $\theta(DF)^2$ either vanish or reduce to θF^3. Further, as $\epsilon^{\mu\nu\rho\sigma}$ does not appear, only two invariants, $\theta^{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ and $\theta^{\mu\nu}F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma}$, are left at the end of the calculation. Note that they are already present in the classical action (2.1).

The procedure to calculate the divergencies in the traces is standard: first we write the traces in the momentum representation, then find their divergent parts by dimensional regularization. An example of this calculation is given in the Appendix. In the result we get:

$$\begin{aligned}\mathrm{Tr}\,[(\Box^{-1}N_1)^2\Box^{-1}T_4] &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x[2(\theta^{\rho\alpha}F^b_{\mu\alpha}+\theta_{\mu\alpha}F^{\rho\alpha b})(A^\mu A_\nu A^\nu A_\rho)_{bc}\\ &+\frac{1}{2}\theta^{\rho\sigma}F^b_{\rho\sigma}(A^\mu A_\nu A^\nu A_\mu)_{bc}],\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,[(\Box^{-1}N_2)^2\Box^{-1}T_2] &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x\left[\frac{3}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\mu A^\mu A_\nu A^\nu)_{bc}\right.\\ &\left.-\,2(\theta^\rho{}_\alpha F^a_{\rho\beta}+\theta^\rho{}_\beta F^a_{\rho\alpha})(F^{\beta\gamma}F_\gamma{}^\alpha)_{bc}+\frac{1}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}(F_{\alpha\gamma}F^{\alpha\gamma})_{bc}\right],\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,[(\Box^{-1}N_1)^4\Box^{-1}T_2] &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x\left[\frac{2}{3}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\alpha A^\alpha A_\beta A^\beta+A_\alpha A_\beta A^\alpha A^\beta\right.\\ &+\,A_\alpha A_\beta A^\beta A^\alpha)_{bc}+(\theta_\rho{}^\alpha F^a_{\sigma\alpha}+\theta_\sigma{}^\alpha F^a_{\rho\alpha})\left(\frac{2}{3}A_\mu A^\mu A^\rho A^\sigma\right.\\ &\left.\left.+\,\frac{2}{3}A_\mu A^\rho A^\mu A^\sigma+\frac{1}{3}A_\mu A^\sigma A^\rho A^\mu+\frac{1}{3}A^\rho A_\mu A^\mu A^\sigma\right)_{bc}\right],\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,[(\Box^{-1}N_1)^3\Box^{-1}T_3] &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x\left[\frac{4}{3}(\theta^{\rho\alpha}F^a_{\sigma\alpha}+\theta_{\sigma\alpha}F^{\rho\alpha a})(A_\rho A^\sigma A^\mu A_\mu\right.\\ &+\,A_\rho A_\mu A^\sigma A^\mu+A_\rho A^\mu A_\mu A^\sigma)_{bc}\\ &\left.+\,\frac{1}{3}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\mu A^\mu A_\nu A^\nu+A_\mu A_\nu A^\mu A^\nu+A_\mu A_\nu A^\nu A^\mu)_{bc}\right],\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,[\Box^{-1}N_2(\Box^{-1}N_1)^2\Box^{-1}T_2 &+ (\Box^{-1}N_1)^2\Box^{-1}N_2\Box^{-1}T_2+\Box^{-1}N_1\Box^{-1}N_2\Box^{-1}N_1\Box^{-1}T_2]\\ &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x\left[\frac{7}{6}\theta^{\rho\sigma}F^a_{\rho\sigma}(2A_\mu A^\mu A_\nu A^\nu\right.\\ &+\,A_\mu A_\nu A^\nu A^\mu)_{bc}\\ &\left.+\,\frac{2}{3}(\theta^{\rho\alpha}F^a_{\sigma\alpha}+\theta_{\sigma\alpha}F^{\rho\alpha a})(A_\rho A_\mu A^\mu A^\sigma+2A_\rho A^\sigma A_\mu A^\mu)_{bc}\right],\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,[\Box^{-1}N_2\Box^{-1}T_4] &= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x[4\theta^{\rho\alpha}F^a_{\mu\alpha}(A^\mu A^\beta A_\beta A_\rho)_{bc}\\ &+\frac{1}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\nu A_\mu A^\mu A^\nu)_{bc}+4i\theta_{\rho\alpha}F^a_{\mu\beta}(A^\mu F^{\alpha\beta}A^\rho)_{bc}\\ &-\,i\theta^{\rho\sigma}F^a_{\rho\sigma}(A^\alpha F_{\alpha\beta}A^\beta)_{bc}-\frac{3N}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}F^b_{\alpha\beta}F^{\alpha\beta c}\end{aligned}$$

JHEP02(2006)046

$$
\begin{aligned}
&- 4i\theta^{\rho\alpha}F^a_{\mu\rho}(A^\mu F_{\alpha\beta}A^\beta)_{bc} + 4i\theta^{\rho\sigma}F^a_{\beta\rho}(A_\alpha F^{\beta\alpha}A_\sigma)_{bc}\\
&- 2i\theta^{\alpha\beta}F^a_{\mu\nu}(A^\nu F_{\alpha\beta}A^\mu)_{ab} - 2i\theta_{\alpha\rho}F^a_{\mu\beta}(A^\rho F^{\beta\alpha}A^\mu)_{bc}\\
&- 2i\theta_{\beta\sigma}F^a_{\mu\alpha}(A^\mu F^{\beta\alpha}A^\sigma)_{bc} + 2i\theta^{\rho\sigma}F^a_{\alpha\beta}(A_\sigma F^{\beta\alpha}A_\rho)_{bc}\\
&+ 2N\theta^{\rho\sigma}F^a_{\alpha\rho}F^b_{\beta\sigma}F^{\alpha\beta c} - 4N\theta^{\rho\beta}F^a_{\mu\rho}F^{\mu\alpha b}F^c_{\beta\alpha}],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Tr}\,(\Box^{-1}N_1\Box^{-1}N_2\Box^{-1}T_3 &+ \Box^{-1}N_2\Box^{-1}N_1\Box^{-1}T_3)\\
&= \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x[\frac{1}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\alpha A^\alpha A_\beta A^\beta\\
&+ A_\mu A_\alpha A^\alpha A^\mu)_{bc} + 2\theta^{\rho\sigma}F^a_{\mu\sigma}(A^\mu A_\rho A_\alpha A^\alpha\\
&+ A_\rho A^\mu A_\alpha A^\alpha + 2A^\mu A_\alpha A^\alpha A_\rho)_{bc}\\
&+ \frac{1}{2}\theta^{\rho\sigma}F^a_{\rho\sigma}(F_{\alpha\beta}F^{\alpha\beta})_{bc} + 4\theta_{\rho\alpha}F^a_{\mu\beta}(F^{\rho\mu}F^{\alpha\beta})_{bc}\\
&- i\theta^{\rho\sigma}F^a_{\rho\sigma}(A^\alpha F_{\alpha\beta}A^\beta)_{bc} + 8i\theta_{\rho\alpha}F^a_{\mu\beta}(A^\mu F^{\alpha\beta}A^\rho)_{bc}\\
&- 2\theta^{\rho\alpha}F^a_{\mu\rho}(F^{\beta\mu}F_{\alpha\beta})_{bc} - 2\theta^{\rho\sigma}F^a_{\beta\rho}(F_{\sigma\alpha}F^{\alpha\beta})_{bc}\\
&+ \theta^{\alpha\beta}F^a_{\rho\nu}(F^{\nu\rho}F_{\alpha\beta})_{bc} + \theta^{\rho\sigma}F^a_{\alpha\beta}(F_{\sigma\rho}F^{\alpha\beta})_{bc}\\
&- 4i\theta^{\rho\alpha}F^a_{\mu\rho}(A^\mu F_{\alpha\beta}A^\beta)_{bc} - 4i\theta^{\rho\sigma}F^a_{\beta\rho}(A_\alpha F^{\alpha\beta}A^\sigma)_{bc}\\
&- 2i\theta^{\alpha\beta}F^a_{\mu\nu}(A^\nu F_{\alpha\beta}A^\mu)_{bc} - 2i\theta^{\rho\sigma}F^a_{\alpha\beta}(A_\sigma F^{\alpha\beta}A_\rho)_{bc}].
\end{aligned}
$$

The difficult part, after summation of particular diagrams, is to transform the sum to the covariant form. We obtain

$$
\Gamma^1|_{\mathrm{div}} = -\frac{11N}{6(4\pi)^2\epsilon}d^{abc}\int d^4x[\frac{1}{4}\theta^{\rho\sigma}F^a_{\rho\sigma}F^b_{\mu\nu}F^{\mu\nu c} - \theta^{\rho\sigma}F^a_{\rho\mu}F^b_{\sigma\nu}F^{\mu\nu c}], \tag{3.7}
$$

where at the end of the calculation we apply the formula from [15], namely

$$
\begin{aligned}
d^{abc}F^a_{\alpha\beta}(F_{\mu\nu}F_{\rho\sigma})_{bc} &= d^{abc}F^a_{\alpha\beta}F^d_{\mu\nu}F^e_{\rho\sigma}(T^dT^e)_{bc}\\
&= F^a_{\alpha\beta}F^d_{\mu\nu}F^e_{\rho\sigma}\,\mathrm{Tr}\,(D^aT^dT^e)\\
&= \frac{N}{2}d^{ade}F^a_{\alpha\beta}F^d_{\mu\nu}F^e_{\rho\sigma},
\end{aligned}
$$

with $(D^a)_{bc} = d^{abc}$. This is the only place where the assumption that fields are in the adjoint representation is used. The full result for the one-loop divergent part of the effective action to first order is

$$
\begin{aligned}
\Gamma_{\mathrm{div}} &= -\frac{1}{4}\left(1 - \frac{22N}{3(4\pi)^2\epsilon}\right)\int d^4x F^a_{\mu\nu}F^{\mu\nu a}\\
&+ \frac{1}{4}\left(1 - \frac{22N}{3(4\pi)^2\epsilon}\right)\theta^{\rho\sigma}d^{abc}\int d^4x\left(\frac{1}{4}F^a_{\rho\sigma}F^b_{\mu\nu}F^{\mu\nu c} - F^a_{\mu\rho}F^b_{\nu\sigma}F^{\mu\nu c}\right).
\end{aligned} \tag{3.8}
$$

From the last equation it is obvious that the one-loop correction to the effective action is proportional to the classical action. The (3.8) can be rewritten in the form

$$
\Gamma_{\mathrm{div}} = -\frac{1}{4}\left(1 - \frac{22N}{3(4\pi)^2\epsilon}\right)\int d^4x F^a_{\mu\nu}\star F^{\mu\nu a}. \tag{3.9}
$$

4. Outlook and conclusions

In the presented calculation the coupling constant was fixed to be 1. However, in order to renormalize the theory, we have to recover it. We take that, initially, the classical lagrangian in $4-\epsilon$ dimensions reads

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{16}g\mu^{\epsilon/2}\theta^{\mu\nu}d^{abc}(F^a_{\mu\nu}F^b_{\rho\sigma}F^{\rho\sigma c} - 4F^a_{\mu\rho}F^b_{\nu\sigma}F^{\rho\sigma c}), \tag{4.1}$$

while the field strength is defined by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\mu^{\epsilon/2}f^{abc}A^b_\mu A^c_\nu. \tag{4.2}$$

The μ is a parameter with the dimension of mass. To cancel divergencies we add counterterms to the initial action. The bare Lagrangian is the sum of the classical Lagrangian and the counterterms; it reads

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L} + \mathcal{L}_{CT} \\ &= -\frac{1}{4}\left(1+\frac{22Ng^2}{3(4\pi)^2\epsilon}\right)F^a_{\mu\nu}F^{\mu\nu a} \\ &+ \frac{1}{4}\left(1+\frac{22Ng^2}{3(4\pi)^2\epsilon}\right)\theta^{\rho\sigma}d^{abc}g\mu^{\epsilon/2}\left(\frac{1}{4}F^a_{\rho\sigma}F^b_{\mu\nu}F^{\mu\nu c} - F^a_{\mu\rho}F^b_{\nu\sigma}F^{\mu\nu c}\right).\end{aligned} \tag{4.3}$$

Introducing the bare quantities

$$\begin{aligned}{A_0}^{\mu a} &= A^{\mu a}\sqrt{1+\frac{22Ng^2}{3(4\pi)^2\epsilon}}, \\ g_0 &= \frac{g\mu^{\epsilon/2}}{\sqrt{1+\frac{22Ng^2}{3(4\pi)^2\epsilon}}},\end{aligned} \tag{4.4}$$

we can rewrite the bare Lagrangian as

$$\mathcal{L}_0 = -\frac{1}{4}{F_0}^a_{\mu\nu}{F_0}^{\mu\nu a} + \frac{1}{16}g_0\theta^{\mu\nu}d^{abc}({F_0}^a_{\mu\nu}{F_0}^b_{\rho\sigma}{F_0}^{\rho\sigma c} - 4{F_0}^a_{\mu\rho}{F_0}^b_{\nu\sigma}{F_0}^{\rho\sigma c}), \tag{4.5}$$

or,

$$\mathcal{L} = -\frac{1}{4}{F_0}^a_{\mu\nu} \star {F_0}^{\mu\nu a}. \tag{4.6}$$

Note that, due to the fact that divergencies of kinetic and θ-linear interaction terms have the same factor, noncommutativity parameter θ need not be renormalized. This rises hope that the theory might be renormalizable to all orders in θ. One can easily find the beta function from (4.4): it is same as in the commutative case

$$\beta = \mu\frac{\partial g}{\partial \mu} = -\frac{11Ng^3}{3(4\pi)^2}. \tag{4.7}$$

The theory is asymptotically free.

Formulae (4.4)–(4.5) mean that the noncommutative pure gauge $SU(N)$ theory is one-loop renormalizable to first order in θ. Divergences can be absorbed in the redefinition of the gauge potential and the gauge coupling constant. The renormalization is standard, multiplicative: no Seiberg-Witten field redefinition is needed, as it was the case in similar calculations [9, 11]. A different result would further strengthen the belief that field theories on noncommutative Minkowski space are not renormalizable. As it is, it opens several possibilities. The first possibility is that the gauge theories are renormalizable in the θ-expanded approach because in this representation the gauge symmetry is introduced in a natural way, via the covariant coordinates. Then fermions are probably inadequately represented, as we know that their presence breaks renormalizability. An obvious further step to check this claim is, for example, to find the second-order divergencies; also, one could consider renormalizability at two loops. The other possible interpretation is that renormalizability is obstructed by the use of Moyal-Weyl $\star$-product [16], and that the more appropriate representation has to be found. In any case, the question deserves further consideration.

Acknowledgments

This work is supported by the Serbian Ministry of Science Grant No. 1486.

A. An example of trace calculation

In this appendix we present the calculation of the diagram $\mathrm{Tr}\,((\Box^{-1}N_1)^4\Box^{-1}T_2)$; other terms in the one-loop effective action are computed in a similar manner.

$$\begin{aligned}
\mathrm{Tr}\,((\Box^{-1}N_1)^4\Box^{-1}T_2) &= \int d^4x d^4y d^4z d^4u d^4v\; g^{\alpha\kappa} G(x-y) N_1{}^{ab}_{\alpha\beta}(y) \\
&\times G(y-z) N_1{}^{\beta\gamma bc}(z) G(z-u) N_1{}^{cd}_{\gamma\delta}(u) \\
&\times G(u-v) N_1{}^{\delta\lambda de}(v) G(v-x) T_2{}^{ea}_{\lambda\kappa}(x) \\
&= 16 d^{abc} \int \frac{d^4x}{(2\pi)^4} \int d^Dp \frac{p^\alpha p^\beta p^\mu p^\nu p^\rho p^\sigma}{(p^2)^5} \\
&\times (A_\alpha A_\beta A_\mu A_\nu)_{bc} (2\theta^\sigma{}_\epsilon F^{\rho\epsilon a} + \frac{1}{4} g^{\rho\sigma}\theta^{\epsilon\lambda} F^a_{\epsilon\lambda}).
\end{aligned} \tag{A.1}$$

All external momenta in vertices vanish, as a consequence of (2.5). The divergent part of the previous integral is found by dimensional regularization. The result is

$$\begin{aligned}
\mathrm{Tr}\,((\Box^{-1}N_1)^4\Box^{-1}T_2) &= d^{abc} \frac{i}{6(4\pi)^2\epsilon} \int d^4x (A_\alpha A_\beta A_\mu A_\nu)_{bc} \\
&\times (2\theta^\sigma{}_\epsilon F^{\rho\epsilon a} + \frac{1}{4} g^{\rho\sigma}\theta^{\epsilon\lambda} F^a_{\epsilon\lambda}) \\
&\times [g_{\alpha\beta}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) \\
&+ g_{\alpha\mu}(g_{\beta\nu}g_{\rho\sigma} + g_{\beta\rho}g_{\nu\sigma} + g_{\beta\sigma}g_{\nu\rho}) \\
&+ g_{\alpha\nu}(g_{\beta\mu}g_{\rho\sigma} + g_{\beta\rho}g_{\mu\sigma} + g_{\beta\sigma}g_{\mu\rho}) \\
&+ g_{\alpha\rho}(g_{\beta\mu}g_{\nu\sigma} + g_{\beta\nu}g_{\mu\sigma} + g_{\beta\sigma}g_{\nu\mu}) \\
&+ g_{\alpha\sigma}(g_{\beta\mu}g_{\nu\rho} + g_{\beta\nu}g_{\mu\rho} + g_{\beta\rho}g_{\mu\nu})] \,.
\end{aligned}$$

From the previous expression we get the final result

$$\begin{aligned}\mathrm{Tr}\,[(\Box^{-1}N_1)^4\Box^{-1}T_2] \;=\; & \frac{i}{(4\pi)^2\epsilon}d^{abc}\int d^4x\left[\frac{2}{3}\theta^{\rho\sigma}F^a_{\rho\sigma}(A_\alpha A^\alpha A_\beta A^\beta + A_\alpha A_\beta A^\alpha A^\beta\right.\\ & +\; A_\alpha A_\beta A^\beta A^\alpha)_{bc} + (\theta_\rho{}^\alpha F^a_{\sigma\alpha} + \theta_\sigma{}^\alpha F^a_{\rho\alpha})\left(\frac{2}{3}A_\mu A^\mu A^\rho A^\sigma\right.\\ & \left.\left.+\;\frac{2}{3}A_\mu A^\rho A^\mu A^\sigma + \frac{1}{3}A_\mu A^\sigma A^\rho A^\mu + \frac{1}{3}A^\rho A_\mu A^\mu A^\sigma\right)_{bc}\right].\end{aligned}$$

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JHEP02(2006)046